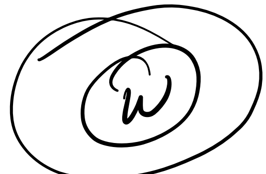


V – vector space

4.2 – Subspaces

Definition: A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem ~~4.2.1~~ Subspace Test
If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.
a) If **u** and **v** are vectors in W, then **u** + **v** is in W.
b) If k is a scalar and **u** is a vector in W, then k**u** is in W.

Since any vector $\vec{w} \in W$ is also in V, vector space axioms 2, 3, 7, 8, 9, 10 are already satisfied. To show $W \subset V$ is itself a vector space, we need to confirm 1, 4, 5, 6. But if 6 holds, then for $\vec{w} \in W$, $k\vec{w} \in \vec{W}$ $\forall k \in \mathbb{R}$. thus, $k=0$ gives $\vec{0} \in W$ and $k=-1$ gives $-\vec{w} \in W$.

As a result, we need only to check that W is closed under addition and scalar multiplication.

An easy way to show W is nonempty is to show $\vec{0} \in W$. (This then has 3 parts.)

Example: (3) Use the Subspace Test to determine which of the sets are subspaces of M_{nn}.

a. The set of all diagonal $n \times n$ matrices.

b. The set of all $n \times n$ matrices A such that $\det(A) = 0$.

c. The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$.

d. The set of all symmetric $n \times n$ matrices.

a) $\vec{0} = \begin{bmatrix} 0 & & 0 \\ & 0 \ddots & \\ 0 & & 0 \end{bmatrix}$ is diagonal so $\vec{0}$ is in the set.

i) Let $\vec{u} = \begin{bmatrix} u_1 & & & 0 \\ & u_2 & & \\ & & \ddots & \\ 0 & & & u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 & & & 0 \\ & v_2 & & \\ & & \ddots & \\ 0 & & & v_n \end{bmatrix}$

then $\vec{u} + \vec{v} = \begin{bmatrix} u_1+v_1 & & & 0 \\ & u_2+v_2 & & \\ & & \ddots & \\ 0 & & & u_n+v_n \end{bmatrix}$, which is diagonal.

$\vec{u} + \vec{v}$ is in the space.

$k\vec{u} = \begin{bmatrix} ku_1 & & & 0 \\ & ku_2 & & \\ & & \ddots & \\ 0 & & & ku_n \end{bmatrix}$, a diagonal matrix.

$k\vec{u}$ is in the space. this is a subspace.

b) Let $\vec{u} = \begin{bmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 \ddots & \\ 0 & & & 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 \ddots & \\ 0 & & & 1 \end{bmatrix}$.

$\det(\vec{u}) = 0$, $\det(\vec{v}) = 0$.

$\vec{u} + \vec{v} = I$ but $\det(I) = 1 \neq 0$.

so $\vec{u}+\vec{v}$ is not in the space.

c) Let $\vec{u}, \vec{v} \in M_{nn}$ such that $tr(u)=tr(v)=0$.

Then $\sum_{i=1}^{n} u_{ii}=0$ and $\sum_{i=1}^{n} v_{ii}=0$.

$tr(u+v)=\sum_{i=1}^{n}(u_{ii}+v_{ii})=\sum_{i=1}^{n}u_{ii}+\sum_{i=1}^{n}v_{ii}=0+0=0$ ✓

$tr(ku)=\sum_{i=1}^{n}(ku_{ii})=k\sum_{i=1}^{n}u_{ii}=k(0)=0$ ✓

This is a subspace.

d) When you have time ☺

set of polynomials degree 3 or less.

Example: (6) Use the Subspace Test to determine which of the sets are subspaces of P_3.

a. All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers.

b. All polynomials of the form $a_0 + a_1x$ in which a_0 and a_1 are real numbers.

a) Let $k = \pi$. No.

b) Let $a_0, a_1, b_0, b_1 \in \mathbb{R}$, $k \in \mathbb{R}$ and

$\vec{p} = a_0 + a_1 x$, $\vec{q} = b_0 + b_1 x$.

$\vec{p} + \vec{q} = (a_0 + b_0) + (a_1 + b_1)x$, with $a_0 + b_0 \in \mathbb{R}$, $a_1 + b_1 \in \mathbb{R}$ ✓

$k\vec{p} = ka_0 + ka_1 x$. $ka_0, ka_1 \in \mathbb{R}$ ✓

yes.

Let A be a 2×2 matrix such that $A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$.

Let B be a 2×2 matrix such that $B\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$.

$$(A+B)\begin{bmatrix}1\\-1\end{bmatrix} = A\begin{bmatrix}1\\-1\end{bmatrix} + B\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}4\\0\end{bmatrix} \neq \begin{bmatrix}2\\0\end{bmatrix}$$

Example: (11) Use the Subspace Test to determine which of the sets are subspaces of M_{22}.

a. All matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$.

b. All matrices of the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$.

$$\begin{bmatrix} a_1 & 1 \\ b_1 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & 1 \\ b_2 & 1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 2 \\ b_1 + b_2 & 2 \end{bmatrix}$$

c. All 2×2 matrices A such that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

a) Let $A = \begin{bmatrix} a_1 & 0 \\ b_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & 0 \\ b_2 & 0 \end{bmatrix}$.

$A + B = \begin{bmatrix} a_1 + a_2 & 0 \\ b_1 + b_2 & 0 \end{bmatrix}$, which has the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$

$kA = \begin{bmatrix} ka_1 & 0 \\ kb_1 & 0 \end{bmatrix}$, which has the same form.

this is a subspace

b) Let $A = \begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$ and $k \in \mathbb{R}$.

$kA = \begin{bmatrix} ak & k \\ bk & k \end{bmatrix}$ if $k \neq 1$, this does not have

structure the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$. No.

c) $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} + B\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Example: (14) Use the Subspace Test to determine which of the sets are subspaces of R^4.

a. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

b. All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where A is as in part (a).

a) Let $k = 0$. Then $kA\vec{x} = 0A\vec{x} = \vec{0}$

and $\vec{0} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

OR Let $\vec{x}_1, \vec{x}_2$ be as described.

closure under addition: Is $\vec{x}_1 + \vec{x}_2$ in the space?

$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

will be a subspace.

Recall: $A\vec{x} = \vec{0}$ is a homogeneous system.

The solution set is $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$

Let $\vec{x}_1, \vec{x}_2$ be solutions to $A\vec{x} = \vec{0}$

$A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0} \Rightarrow \vec{x}_1+\vec{x}_2$ is a solution. $k \in R \Rightarrow A(k\vec{x}_1) = kA\vec{x}_1 = k(\vec{0}) = \vec{0}$

So $k\vec{x}_1$ is a solution.

Theorem 4.2.3 The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

Definition: The solution set of a homogeneous system in n unknowns is a subspace of R^n, called the **solution space** of the system. $A\vec{x} = \vec{0}$

$T(\vec{x}) = \vec{0}$

Definition: Let $T_A : R^n \to R^m$ be multiplication by the coefficient matrix A. The solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n that T_A maps into the zero vector in R^m. This set is called the **kernel** of the transformation.

The kernel is the set of domain elements that the transformation maps to $\vec{0}$ in the codomain.

Theorem 4.2.4 If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of R^n.

Theorem 4.2.2 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

pf: Let $W_1, W_2, \ldots, W_r$ be subspaces of V.

Then $\vec{0} \in \vec{W}_1, \vec{0} \in \vec{W}_2, \ldots, \vec{0} \in \vec{W}_r$

$\Rightarrow \vec{0} \in \vec{W}_1 \cap \vec{W}_2 \cap \cdots \cap \vec{W}_r$.

Let $\vec{u}, \vec{v} \in \vec{W}_1 \cap \vec{W}_2 \cap \cdots \cap \vec{W}_r \Rightarrow \vec{u}, \vec{v} \in W_1, \vec{u}, \vec{v} \in W_2, \ldots$ $\vec{u}, \vec{v} \in \vec{W}_r$. Since W_i are subspaces, $\vec{u}+\vec{v} \in W_1, \vec{u}+\vec{v} \in W_2$ $\ldots, \vec{u}+\vec{v} \in W_r \Rightarrow \vec{u}+\vec{v} = W_1 \cap W_2 \cap \cdots \cap W_r$.

$k\vec{u} \in W_1, W_2, \ldots, W_r$ (subspaces) $\Rightarrow k\vec{u} \in \bigcap_{i=1}^{r} W_i$ so

Subspace test: i) if $\vec{u}, \vec{v} \in W$, then $\vec{u}+\vec{v} \in W$

ii) If $\vec{u} \in W, k \in R$, then $k\vec{u} \in W$

$W_1 \cap W_2 \cap \cdots \cap W_r$ is a subspace.

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\} = \{\vec{v}_i\}_{i=1}^{r}$

Examples of Subspaces

$\{\mathbf{0}\}$ is a subspace of every vector space V

Any vector space V is a subspace of itself

Subspace of R^2

Lines through the origin

Subspaces of R^3

Lines through the origin

Planes through the origin

The solution space of a homogeneous system in n unknowns is a subspace of R^n

Subspaces of M_{nn}

Symmetric matrices

Triangular matrices

Diagonal matrices

Subspaces of $F(-\infty, \infty)$ (The following is actually a sequence of nested subspaces)

$C(-\infty, \infty)$, the set of functions continuous on R

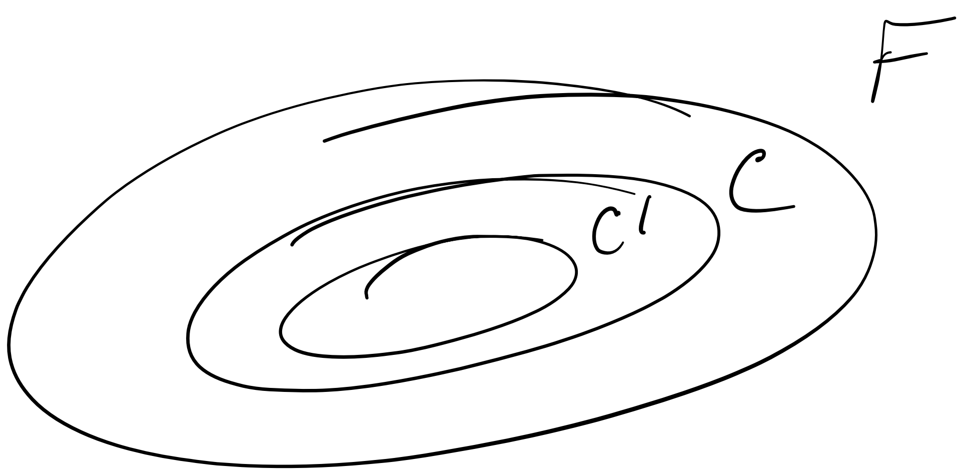
$C^1(-\infty, \infty)$, the set of functions with continuous first-order derivatives on R

$C^n(-\infty, \infty)$, the set of functions with continuous n^{th}-order derivatives on R

$C^\infty(-\infty, \infty)$, the set of functions with derivatives of all orders on R

P_∞, the set of polynomials

P_n, polynomials of degree $\leq n$



Consider

$$W = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \text{ for } A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \}.$$

Let $\vec{x}_1, \vec{x}_2 \in W$. Is $\vec{x}_1 + \vec{x}_2 \in W$?

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$$ closed under addition ✓

$$A(k\vec{x}_1) = kA\vec{x}_1 = k\vec{0} = \vec{0}$$ closed under scalar mult. ✓

But the set is empty and so is not a subspace.

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ is not defined.